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PAPPUS TYPE THEOREMS FOR HYPERSURFACES IN A SPACE FORM*

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ABSTRACT

In order to get further insight on the Weyl's formula for the volume of a tubular hypersurface, we consider the following situation. Let $c(t)$ be a curve in a space form M^n_λ of sectional curvature λ. Let P_0 be a totally geodesic hypersurface of M^n_λ through $c(0)$ and orthogonal to $c(t)$. Let $\mathcal{C}_0$ be a hypersurface of P_0. Let $\mathcal{C}$ be the hypersurface of M^n_λ obtained by a motion of $\mathcal{C}_0$ along $c(t)$. We shall denote it by $\mathcal{C}^P$ or $\mathcal{C}^F$ if it is obtained by a parallel or Frenet motion, respectively. We get a formula for volume($\mathcal{C}$). Among other consequences of this formula we get that, if $c(0)$ is the centre of mass of $\mathcal{C}_0$, then volume($\mathcal{C}$) $\geq$ volume($\mathcal{C}^P$), and the equality holds when $\mathcal{C}_0$ is contained in a geodesic sphere or the motion corresponds to a curve contained in a hyperplane of the Lie algebra $\mathcal{O}(n-1)$ (when $n=3$, the only motion with these properties is the parallel motion).

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§1. Introduction

The Weyl's formulae for the volumes of a tube around a submanifold P in $\mathbb{R}^n$ and its correspondent boundary reveal the remarkable qualitative fact that these volumes depend only on the intrinsic geometry of P and the radius of the tube (cf. [We] and [Gr1, page 1]). In [GM], A. Gray and the third author have begun the work of understanding the deeper reasons of this fact. They have done it completely for the volume of a tube around a curve $c(t)$ in a simply connected space form M^n_λ of dimension n and constant sectional curvature λ. In this case, given a domain $\mathcal{D}_0$ in a totally geodesic hypersurface P_0 containing $c(0)$ and orthogonal to the curve $c(t)$, they have got a formula for the volume of the domain $\mathcal{D}$ in M^n_λ obtained by a motion of $\mathcal{D}_0$ along the curve $c(t)$ (generalizing a classical Pappus formula for the volume, see [GG]). As a consequence of this formula, if $c(0)$ is the centre of mass of $\mathcal{D}_0$, then volume($\mathcal{D}$) depends only on $\mathcal{D}_0$ and the length of c, like in the Weyl's formula for the volume of a tube. This means that, for volumes of tubes around curves in M^n_λ, the only fact that really matters for Weyl's formula is that any section of the tube by a geodesic hyperplane orthogonal to the curve has its centre of mass on the curve. It is not necessary that this section be a geodesic ball of P_0, like in tubes.

To understand the volume of a tubular hypersurface (the boundary of a tube) we consider a connected hypersurface $\mathcal{C}_0$ of P_0 with its centre of mass at $c(0)$ and compute the volume of the hypersurface $\mathcal{C}$ obtained by a motion of $\mathcal{C}_0$ along the curve $c(t)$. Goodman and Goodman ([GG]) have shown that, for curves in $\mathbb{R}^3$, the spherical shape of $\mathcal{C}_0$ is important. In fact, they show with examples that, when $\mathcal{C}_0$ is not a circle, the area of $\mathcal{C}$ depends on the curvature and the torsion of the curve. However, in [GM] it is shown (for a general M^n_λ) that the role of this spherical shape can be overcome when the motion along the curve is parallel. In this case, the authors of [GM] have obtained a formula for volume($\mathcal{C}$) completely similar to the formula for volume($\mathcal{D}$). Nevertheless, the role of parallel motion among all the motions is not completely understood, and this is lacking for the full comprehension of Weyl's formula for the volume of a tubular hypersurface around a curve in M^n_λ.

The aim of this paper is to fill in this lack by achieving a full understanding of the role of parallel motion. Concretely, we shall see that:

(a) Parallel motion gives the lowest value for the volume of a hypersurface obtained by a motion along a curve (Theorem 4.1). This result was suggested to the authors by figure 3 in [GM]. It can also give a feeling of this result to the reader.

When the hypersurface $\mathcal{C}_0$ is connected, compact and without boundary, it encloses a domain $\mathcal{D}_0$; then $\mathcal{C}$ is the boundary (with the exception of the "top and the bottom layers") of the domain $\mathcal{D}$ obtained by the motion of $\mathcal{D}_0$ along $c(t)$. When $c(0)$ is also the centre of mass of $\mathcal{D}_0$, it follows from [GM, Th. 1] that volume($\mathcal{D}$) is the same for all the motions. Then our result says that parallel motion encloses the same volume and gives the minimum area, and it can be interpreted as an "isoperimetric inequality".

(b) For any curve, and for a generic motion along the curve (then, not parallel) the volume of $\mathcal{C}$ depends only on the length of the curve and on $\mathcal{C}_0$ if and only if $\mathcal{C}_0$ is part of a geodesic sphere (Theorem 4.4). In the special case of dimension $n = 3$, this holds for every non-parallel motion.

Another look at the same results: In dimension 3, if $\mathcal{C}_0$ is not a circle or a circle without one point, the parallel motion is the unique where volume($\mathcal{C}$) attains its minimum. For greater dimensions, the motion giving the minimum is not unique for some $\mathcal{C}_0$ not contained in a geodesic sphere, but the set of motions where the minimum is not attained is open and dense in the set of motions along with an appropriate topology.

(c) For a Frenet motion along a generic curve, the volume of $\mathcal{C}$ depends only on the length of the curve and on $\mathcal{C}_0$ if and only if $\mathcal{C}_0$ is as in (b)(Theorem 5.1).

The key point for getting these results is a formula for volume($\mathcal{C}$) valid for a general motion. We shall get it in section 3. It will show that, in general, volume($\mathcal{C}$) depends on all the curvatures of $c(t)$, in contrast to the general expression for volume($\mathcal{D}$). The formula in [GM] for volume($\mathcal{C}$) when $\mathcal{C}$ is obtained by a parallel motion, and the precise statements of (a), (b) and (c) will be a (nontrivial) consequence of this general formula. These statements will be given in sections 4 and 5. In the next one, we shall collect some definitions.

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§2. Preliminaries

Throughout this paper, $c : I = [0, L] \longrightarrow M^n_\lambda$ will denote a C^∞ curve parametrized by its arc-length t. We shall suppose that c is an embedding from $[0, L]$ into M^n_λ if $c(0) \neq c(L)$ or induces an embedding from S^1 into M^n_λ if $c(0) = c(L)$.

We say that **an orthonormal frame** $\{e_1(t) = c'(t), e_2(t), \ldots, e_n(t)\}$ **is**

parallel if

$$\frac{D}{dt}e_j(t) = 0, \quad \text{for } 2 \leq j \leq n,$$

where D is the normal connection on the normal bundle of c (that is, $\frac{D}{dt}e_j(t)$ is the component of $\frac{\nabla}{dt}e_j(t)$ normal to $c'(t)$).

For every $t \in [0, L]$, let P_t be the complete totally geodesic hypersurface of M^n_λ through $c(t)$ and orthogonal to the curve c. It will be called the **geodesic hyperplane through** $c(t)$. In this paper, any totally geodesic submanifold of M^n_λ will be called a **geodesic subspace.**

A **motion along** c **associated to a smooth orthonormal frame** $\{E_1(t) = c'(t), E_2(t), \ldots, E_n(t)\}$ **along** $c(t)$ is the family $\Phi := \{\phi_t\colon P_0 \to P_t\}_{t\in[0,L]}$ of diffeomorphisms defined by

$$\phi_t\left(\exp_{c(0)} \sum_{i=2}^{n} \mu^i E_i(0)\right) = \exp_{c(t)} \sum_{i=2}^{n} \mu^i E_i(t). \tag{2.1}$$

From this definition it follows that $\phi_{t*c(0)}E_i(0) = E_i(t)$; then

$$\varphi_t := \phi_{t*c(0)}\colon T_{c(0)}P_0 \longrightarrow T_{c(t)}P_t \tag{2.2}$$

is an isometry and, for every $\mu \in \{c'(0)\}^\perp$, $\phi_t(\exp_{c(0)} \mu) = \exp_{c(t)} \varphi_t\mu$.

Moreover, it follows also from (2.1) that ϕ_t takes geodesics of P_0 through $c(0)$ into geodesics of P_t through $c(t)$. Then, since $\phi_{t*c(0)}$ is an isometry and P_0 and P_t are space forms with the same sectional curvature λ, it follows from Cartan's Theorem ([dC, page 156]) that $\phi_t\colon P_0 \longrightarrow P_t$ **is an isometry**.

A curve $c(t)$ in M^n_λ with $\{c'(t),, c''(t), \ldots, c^{(n-1)}(t)\}$ linearly independent at every t has a unique Frenet frame $\{f_1(t) = c'(t), f_2(t), \ldots, f_n(t)\}$, satisfying the well known Frenet equations (cf. [GM] or [Sp]). Many curves that do not satisfy this property have still a (not unique) frame satisfying the Frenet equations, and it will also be called a Frenet frame. On this paper we shall **suppose that all the curves that we consider have a Frenet frame**.

A **Frenet motion** is a motion associated to a Frenet frame of c (according to the above remark, it is unique only on curves not contained in any geodesic hyperplane). It will be denoted by $\Phi^F = \{\phi^F_t\}$, and $\varphi^F_t = \phi^F_{t*c(0)}$.

A **parallel motion** is a motion associated to a parallel frame along c; it is unique along any given curve. It will be denoted by $\Phi^P = \{\phi^P_t\}$, and $\varphi^P_t = \phi^P_{t*c(0)}$.

The symbol ($'$) will denote the usual derivative for a function from $\mathbb{R}$ to $\mathbb{R}$, the tangent vector for a curve in M^n_λ, and the covariant derivative for a vector field along a curve.

For every $\lambda \in \mathbb{R}$, $s_\lambda \colon \mathbb{R} \to \mathbb{R}$ will denote the solution of the equation $s''+\lambda s = 0$ with the initial conditions $s(0) = 0$ and $s'(0) = 1$; and $c_\lambda = s'_\lambda$.

Let Γ be an oriented totally geodesic hypersurface of M_λ^{n-1} with unit normal vector field ζ; let $o \in \Gamma$. For every $x \in M_\lambda^{n-1}$, let γ_x be the unique minimizing geodesic joining o and x, with $\gamma_x(0) = o$. Let $r \colon M_\lambda^{n-1} \longrightarrow \mathbb{R}$ be the function defined by $r(x) = \operatorname{dist}(o, x)$.

Given a submanifold $\mathcal{B}$ of M_λ^{n-1}, with compact closure, we define **the moment $M_\Gamma(\mathcal{B})$ of $\mathcal{B}$ respect to Γ** by the integral

$$M_\Gamma(\mathcal{B}) = \int_{\mathcal{B}} s_\lambda(r(x)) \langle \gamma'_x(0), \zeta_o \rangle \sigma, \tag{2.3}$$

where σ is the volume element of $\mathcal{B}$. Elementary formulae for geodesic triangles prove that $M_\Gamma(\mathcal{B})$ does not depend on o (cf. [GM])

We say that a point $o \in M_\lambda^{n-1}$ is **the centre of mass of $\mathcal{B}$** if $M_\Gamma(\mathcal{B}) = 0$ for every oriented totally geodesic hypersurface Γ through o. This definition coincides with that given in [He], but it is slightly different from the usual one (cf. [Ka] or [BK]), where the centre of mass of $\mathcal{B}$ is the point where the function $F \colon p \mapsto \int_{\mathcal{B}} \operatorname{dist}(p, x)^2 \, dx$ attains its minimum. This is also true for our definition if $\lambda = 0$. However, if $\lambda \neq 0$, the function F has to be changed by

$$\mathcal{F} \colon p \mapsto -\lambda \int_{\mathcal{B}} c_\lambda(\operatorname{dist}(p, x)) \, dx.$$

With this small change, the arguments in [Ka] or [BK] to prove the existence and uniqueness of the centre of mass (with $\mathcal{B}$ contained in a ball of radius $\leq \pi/4\sqrt{\lambda}$ if $\lambda > 0$) still work here.

Given the motion $\Phi = \{\phi_t \colon P_0 \longrightarrow P_t\}_{t \in [0,L]}$ along c associated to $\{E_i(t)\}_{i=1}^n$, we shall denote by U_0 an open set of P_0 such that $U = \bigcup_{t \in I} \phi_t(U_0)$ is the image by exp of an open set of the normal bundle of c on which exp is a diffeomorphism.

From now on,

$\mathcal{C}_0$ will denote a **connected and embedded hypersurface** of P_0 with compact closure, and satisfying $\mathcal{C}_0 \subset U_0$;

for any motion $\{\phi_t\}$, $\mathcal{C}_t = \phi_t(\mathcal{C}_0)$ and $\mathcal{C} = \bigcup_{t \in [0,L]} \mathcal{C}_t$. $\mathcal{C}$ is called **the hypersurface obtained by the motion $\{\phi_t\}$ of $\mathcal{C}_0$ along** c;

$\mathcal{C}^F$ and $\mathcal{C}^P$ will denote, respectively, the hypersurfaces obtained by a Frenet or parallel motion of $\mathcal{C}_0$ along c.

It will be clear from the proofs that our main theorems will also be true for immersed hypersurfaces.

§3. The main formula

Before we state and prove the main formula we shall recall that the cross vector product of $n-1$ vectors $X_1, \ldots, X_{n-1}$ in an oriented Riemannian manifold M of dimension n with volume form ω is given by

$$\langle X_1 \wedge \ldots \wedge X_{n-1}, u\rangle = \omega(X_1, \ldots, X_{n-1}, u) \quad \text{for any vector } u.$$

Given a point $x_0 \in \mathcal{C}_0$, and a motion Φ along $c(t)$, we shall use the following notation:

$x_t = \phi_t(x_0) \in \mathcal{C}_t$;
γ_{x_t} is the unique minimizing geodesic joining $c(t)$ and x_t, with $\gamma_{x_t}(0) = c(t)$;
$N(t) = \gamma'_{x_t}(0)$ is its tangent vector at $c(t)$;
τ_t is the parallel transport along γ_{x_t} from $c(t)$ to x_t,
ξ_t is the unit vector in $T_{x_t}P_t$ orthogonal to $\mathcal{C}_t$ (then $\xi_t = \phi_{t*x_0}\xi_0$);
$N_i(t) = \langle N(t), f_i(t)\rangle$; and
$r(x_t) = \operatorname{dist}(c(t), x_t)$ (then $\gamma_{x_t}(r(x_t)) = x_t$).

THEOREM 3.1:

volume$(\mathcal{C}) =$

$$\int_0^L \left(\int_{\mathcal{C}_t} \sqrt{\langle \tau_t \frac{DN}{dt}(t), \xi_t\rangle^2 s_\lambda(r)^2 + (c_\lambda(r) - s_\lambda(r)N_2(t)k_1(t))^2}\ \eta_t \right) dt,$$

where η_t is the volume element of $\mathcal{C}_t$.

Proof: Let $\psi\colon]0, L[\times\mathcal{C}_0 \longrightarrow \mathcal{C}$ be the diffeomorphism defined by

$$\psi(t, x_0) = \exp_{c(t)} \varphi_t(\mu) \quad \text{with } x_0 = \exp_{c(0)} \mu.$$

Let $\{e_3, \ldots, e_n\}$ be an orthonormal basis of $T_{x_0}\mathcal{C}_0$ and let η_0 and η be the volume elements of $\mathcal{C}_0$ and $\mathcal{C}$, respectively. Using the properties of the cross vector product,

$$\begin{aligned}\psi^*\eta = \psi^*\eta\Big(\frac{\partial}{\partial t}, e_3, \ldots, e_n\Big)dt \wedge \eta_0 &= \eta\Big(\psi_*\frac{\partial}{\partial t}, \psi_* e_3, \ldots, \psi_* e_n\Big)dt \wedge \eta_0\\ &= \Big|\psi_*\frac{\partial}{\partial t} \wedge \psi_* e_3 \wedge \ldots \wedge \psi_* e_n\Big| dt \wedge \eta_0.\end{aligned}$$

Then the volume of $\mathcal{C}$ is

$$\begin{aligned}\text{volume}(\mathcal{C}) &= \int_{\mathcal{C}} \eta = \int_0^L \int_{\mathcal{C}_0} \psi^*(\eta)\\ &= \int_0^L \int_{\mathcal{C}_0} \Big|\psi_*\frac{\partial}{\partial t} \wedge \psi_* e_3 \wedge \ldots \wedge \psi_* e_n\Big|\ \eta_0\ dt.\end{aligned} \tag{3.1}$$

To compute the integrand of (3.1), first we observe that

$$\psi_* \frac{\partial}{\partial t} = \frac{d}{dt}\psi(t, \exp_{c(0)} \mu) = \frac{d}{dt} \exp_{c(t)} \varphi_t(\mu)$$

is the Jacobi field Y_1 along γ_{x_t} at $r = |\mu|$, computed in [GM, (13)], given by

$$\psi_* \frac{\partial}{\partial t} = Y_1(r) = c_\lambda(r)\tau_t f_1(t) + s_\lambda(r)\tau_t \left(\left\langle \frac{\nabla N}{dt}, f_1\right\rangle f_1 + \frac{DN}{dt}\right). \tag{3.2}$$

Let $c_i(s)$ be a curve in $\mathcal{C}_0$ such that $c_i(0) = p$, $c_i'(0) = e_i$; then

$$\psi_* e_i = \left.\frac{d}{ds}\right|_{s=0} \psi(t, c_i(s)) = \left.\frac{d}{ds}\right|_{s=0} \phi_t(c_i(s)) = \phi_{t*c_i(0)} e_i =: \overline{e_i} \tag{3.3}$$

with $\{\overline{e_3}, \ldots, \overline{e_n}\}$ an orthonormal basis of $T_{\phi_t(x_0)}\mathcal{C}_t$, since ϕ_t is an isometry. Moreover, using the cross vector product in P_t, we have

$$\xi_t = \overline{e_3} \wedge \cdots \wedge \overline{e_n} \in T_{x_t} P_t. \tag{3.4}$$

Therefore, from the expressions (3.2) and (3.3), we obtain

$$\psi_* \frac{\partial}{\partial t} \wedge \psi_* e_3 \wedge \cdots \wedge \psi_* e_n = Y_1 \wedge \overline{e_3} \wedge \cdots \wedge \overline{e_n}. \tag{3.5}$$

Furthermore, $T_{c(t)}P_t$ is generated by $\{f_2(t), \ldots, f_n(t)\}$; then $T_{x_t}P_t$ is generated by $\{\tau_t f_2(t) = \overline{f_2}(t), \ldots, \tau_t f_n(t) = \overline{f_n}(t)\}$, which is a positively oriented orthonormal basis. We shall use it to compute the cross vector product in P_t with formula (3.5). We shall denote by $\overline{e}_i^j$, $2 \leq j \leq n$ the components of $\overline{e}_i$ in this basis.

Using the expression (3.2) for $Y_1(r)$, and the basis $\{\overline{f_1}(t), \overline{f_2}(t), \ldots, \overline{f_n}(t)\}$ to compute the cross vector product in M_λ^n, we get

$$\begin{aligned}
&Y_1 \wedge \overline{e_3} \wedge \cdots \wedge \overline{e_n} \\
&= (-1)^{n-1} \begin{vmatrix} \overline{f_1} & \overline{f_2} & \cdots & \overline{f_n} \\ c_\lambda(r) + s_\lambda(r)\langle \tau_t \frac{\nabla N}{dt}, \overline{f_1}\rangle & s_\lambda(r)\langle \tau_t \frac{DN}{dt}, \overline{f_2}\rangle & \cdots & s_\lambda(r)\langle \tau_t \frac{DN}{dt}, \overline{f_n}\rangle \\ 0 & \overline{e_3}^2 & \cdots & \overline{e_3}^n \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \overline{e_n}^2 & \cdots & \overline{e_n}^n \end{vmatrix} \\
&= (-1)^{n-1} \begin{vmatrix} \langle \tau_t \frac{DN}{dt}, \overline{f_2}\rangle & \cdots & \langle \tau_t \frac{DN}{dt}, \overline{f_n}\rangle \\ \overline{e_3}^2 & \cdots & \overline{e_3}^n \\ \cdots & \cdots & \cdots \\ \overline{e_n}^2 & \cdots & \overline{e_n}^n \end{vmatrix} s_\lambda(r)\, \overline{f_1}. \\
&\quad + (-1)^{n-2}(c_\lambda(r) + s_\lambda(r)\langle \tau_t \frac{\nabla N}{dt}, \overline{f_1}\rangle) \begin{vmatrix} \overline{f_2} & \cdots & \overline{f_n} \\ \overline{e_3}^2 & \cdots & \overline{e_3}^n \\ \cdots & \cdots & \cdots \\ \overline{e_n}^2 & \cdots & \overline{e_n}^n \end{vmatrix}
\end{aligned}$$

$$= -(-1)^{n-2}\left\langle \tau_t \frac{DN}{dt}, \begin{vmatrix} \overline{f_2} & \cdots & \overline{f_n} \\ \overline{e_3}^2 & \cdots & \overline{e_3}^n \\ \cdots & \cdots & \cdots \\ \overline{e_n}^2 & \cdots & \overline{e_n}^n \end{vmatrix} \right\rangle s_\lambda(r)\, \overline{f_1}$$
$$+ \left(c_\lambda(r) + \langle \tau_t \frac{\nabla N}{dt}, \overline{f_1} \rangle s_\lambda(r) \right) \overline{e_3} \wedge \cdots \wedge \overline{e_n}.$$

Then, recalling (3.4) and using

$$\left\langle \tau_t \frac{\nabla N}{dt}, \overline{f_1} \right\rangle = \left\langle \frac{\nabla N}{dt}, f_1 \right\rangle = -\left\langle N, \frac{\nabla f_1}{dt} \right\rangle = -\langle N, k_1 f_1 \rangle - k_1 \langle N, f_2 \rangle = -k_1 N_2,$$

we obtain

$$Y_1 \wedge \overline{e_3} \wedge \cdots \wedge \overline{e_n} = -\left\langle \tau_t \frac{DN}{dt}, \xi_t \right\rangle s_\lambda(r)\, \overline{f_1} + (c_\lambda(r) - s_\lambda(r) N_2(t) k_1(t)) \xi_t. \tag{3.6}$$

So, if we substitute (3.6) in (3.1), we obtain

$$\begin{aligned} \mathrm{volume}(\mathcal{C}) &= \int_0^L \int_{\mathcal{C}_0} |Y_1 \wedge \overline{e_3} \wedge \cdots \wedge \overline{e_n}|\, \eta_0 \quad dt \\ &= \int_0^L \left(\int_{\mathcal{C}_0} \sqrt{\langle \tau_t \frac{DN}{dt}(t), \xi_t \rangle^2 s_\lambda(r)^2 + (c_\lambda(r) - s_\lambda(r) N_2(t) k_1(t))^2}\; \eta_0 \right) dt; \end{aligned}$$

and the formula of the theorem follows taking into account that the ϕ_t are isometries. ∎

Remark: With the convention that $N_j = 0 = k_j$ if $j \notin \{1, \ldots, n\}$, $N_1 = 0$ and $k_n = 0$, in [GM] the following formula is given:

$$\frac{DN}{dt}(t) = \sum_{i=2}^{n} (N_i' - N_{i+1} k_i + N_{i-1} k_{i-1})(t) f_i(t). \tag{3.7}$$

Then, in general, all the curvatures of c appear in the formula for volume($\mathcal{C}$), a situation very different from that of the volume of a domain. This dependence is real, and not a defect of the formula, as can be checked taking helices in $\mathbb{R}^3$ with the same curvature and different torsion.

§4. The role of motions

THEOREM 4.1: *Let $\mathcal{C}_0$ be a hypersurface with centre of mass at $c(0)$. Then*

$$\mathrm{volume}(\mathcal{C}) \geq \mathrm{volume}(\mathcal{C}^P) = L \int_{\mathcal{C}_0} c_\lambda(r)\, \eta_0. \tag{4.1}$$

Proof: Since ϕ_t is an isometry,

$$\int_{\mathcal{C}_t} c_\lambda(r)\ \eta_t = \int_{\mathcal{C}_0} c_\lambda(r)\ \eta_0.$$

Let us denote by $M_{\Gamma_t}(\mathcal{C}_t)$ the moment of $\mathcal{C}_t$ with respect to the geodesic hyperplane Γ_t of P_t through $c(t)$ orthogonal to $f_2(t)$. From Theorem 3.1 and the definition (2.3), it follows that

$$\begin{aligned} \text{volume}(\mathcal{C}) &\geq \int_0^L \left\{ \int_{\mathcal{C}_t} c_\lambda(r)\ \eta_t - \int_{\mathcal{C}_t} s_\lambda(r) N_2(t)\ k_1(t)\ \eta_t \right\} dt \\ &= L \int_{\mathcal{C}_0} c_\lambda(r)\ \eta_0 - \int_0^L M_{\Gamma_t}(\mathcal{C}_t)\ k_1(t)\ dt. \end{aligned} \tag{4.2}$$

But, since ϕ_t is an isometry and $c(0)$ is the centre of mass of $\mathcal{C}_0$, then $c(t)$ is the centre of mass of $\mathcal{C}_t = \phi_t(\mathcal{C}_0)$, and $M_{\Gamma_t}(\mathcal{C}_t) = 0$. On the other hand, it is obvious that (4.2) is an equality for a parallel motion, which gives the equality in (4.1). Then, the inequality in (4.1) follows. ∎

After proving Theorem 4.1, a problem of uniqueness arises: Is parallel motion the only one giving the minimum of volume($\mathcal{C}$) for a given $\mathcal{C}_0$? If $\mathcal{C}_0$ is a geodesic sphere of P_0, then $\mathcal{C}$ is a tube around $c(t)$, and Weyl's tube formula says that all motions give the same value for volume($\mathcal{C}$); then the above question has to be modified by restricting $\mathcal{C}_0$ not to be a geodesic sphere. Another viewpoint of the same question is the following:

When we look at the proof of Weyl's formula for the volume of a tubular hypersurface, a prominent role is played by the fact that we have to integrate along the spheres which are the normal section of the tube. Theorem 4.1 says that this role disappears when we consider parallel motions along a curve. Is this the unique motion producing this phenomenon?

In the next theorem we shall give an answer to these questions. We shall see that parallel motion is unique with the above properties when $n = 3$ and that, for $n \geq 4$, the special role of spherical sections is played on a generic motion.

To state the theorem we shall need two lemmas. The first is well known and we shall omit the proof. The second is the crucial technical remark from which we shall get the results.

LEMMA 4.2: *If $\gamma'_{x_0}(r(x_0)) = \pm\xi_0$ for every $x_0 \in \mathcal{C}_0$, then $\mathcal{C}_0$ is contained in a geodesic sphere of P_0 with centre at $c(0)$.*

If $n = 3$, then $\mathcal{C}_0$ is a geodesic circle, perhaps without one point. If $\mathcal{C}_0$ is compact without boundary, then, for any dimension $n \geq 3$, $\mathcal{C}_0$ is a geodesic sphere of P_0.

In the next lemma and other results, we shall usually conclude that $\mathcal{C}_0$ (if it is not closed) is a circle perhaps without one point in case $n = 3$, but for $n > 3$ we shall only conclude that $\mathcal{C}_0$ is contained in a sphere. The reason for this is that a connected subset of a circle has a centre of mass which coincides with the centre of the circle only if the complement is at most one point, while for higher dimensions there are many different subsets with their centre of mass at the centre of the sphere. These subsets may not even be centrally symmetric. The simplest ones are tubes around suitable pieces of totally geodesic submanifolds.

LEMMA 4.3: *Let $\mathcal{C}_0$ be a hypersurface with centre of mass at $c(0)$. Let $\Phi := \{\phi_t\}_{t\in[0,L]}$ be a motion along c such that, for every $x_0 \in \mathcal{C}_0$ (i.e., for every $N(0) = \gamma'_{x_0}(0)$), there are $n-2$ points $t_2, \ldots, t_{n-1}$ such that*

$$\text{(4.3)} \qquad \text{the vectors} \quad \varphi_{t_i}^{-1}\frac{DN}{dt}(t_i),\ 2 \le i \le n-1, \quad \text{are linearly independent.}$$

If volume$(\mathcal{C})$ = volume$(\mathcal{C}^P)$, *then $\mathcal{C}_0$ is contained in a geodesic sphere of P_0 with centre at $c(0)$.*

If $n = 3$, then $\mathcal{C}_0$ is a geodesic circle, perhaps without one point. If $\mathcal{C}_0$ is compact without boundary, then, for any dimension $n \ge 3$, $\mathcal{C}_0$ is a geodesic sphere of P_0.

Proof: From the proof of 4.1, it is obvious that the equality volume$(\mathcal{C})$ = volume$(\mathcal{C}^P)$ holds if and only if

$$\text{(4.4)} \qquad \left\langle \tau_t \frac{DN}{dt}(t), \xi_t \right\rangle = 0.$$

Since ϕ_t are isometries, we have

$$\left\langle \phi_{t_i*x_0}^{-1} \tau_{t_i} \frac{DN}{dt}(t_i), \xi_0 \right\rangle = 0 \quad \text{for } 2 \le i \le n-1.$$

From this equality, the hypothesis on the motion, and from the facts that

$$\left\langle \phi_{t_i*x_0}^{-1} \tau_{t_i} \frac{DN}{dt}(t_i), \gamma'_{x_0}(\mathrm{dist}(x_0, c(0))) \right\rangle = \left\langle \frac{DN}{dt}(t_i), N(t_i) \right\rangle = 0$$
$$\text{and} \quad \phi_{t_i*x_0}^{-1} \tau_{t_i} \frac{DN}{dt}(t_i) \in T_{x_0}P_0,$$

we get that $\gamma'_{x_0}(\mathrm{dist}(x_0, c(0))) = \pm\xi_0$, and the thesis follows from Lemma 4.2. ∎

Now we make some remarks about the motion along a curve.

Recall that Φ^P denotes the parallel motion along a curve c in M_λ^n. Given any motion Φ along c, we consider the maps

$$A(t) = (\varphi_t^P)^{-1} \circ \varphi_t \colon T_{c(0)}P_0 \longrightarrow T_{c(0)}P_0,$$

which define a C^∞-curve $t \mapsto A(t)$ in the Lie group $\mathrm{SO}(n-1)$ of isometries of $T_{c(0)}P_0$ preserving the orientation, because φ_t^P and φ_t are isometries and $A(0) = \mathrm{Id}$. This allows us to identify the motions along $c\colon I \longrightarrow M_\lambda^n$ with the curves $A\colon I \longrightarrow \mathrm{SO}(n-1)$. Then we can give an interpretation of the condition (4.3) in terms of the curves $A(t)$.

A simple computation shows that

$$\varphi_t^{-1}\frac{DN}{dt}(t) = \varphi_t^{-1} \circ \varphi_t^P \circ (\varphi_t^P)^{-1}\frac{DN}{dt}(t) = A^{-1}(t)A'(t)N(0).$$

Then, condition (4.3) is equivalent to

$$\text{(4.5)} \qquad A^{-1}(t_i)A'(t_i)N(0), \quad 2 \leq i \leq n-1, \quad \text{are linearly independent.}$$

Moreover, on a neighbourhood of $\mathrm{Id} = A(0) \in \mathrm{SO}(n-1)$, the inverse of the exponential map $\ln\colon \mathrm{SO}(n-1) \longrightarrow \mathcal{O}(n-1)$ from $\mathrm{SO}(n-1)$ to its Lie algebra $\mathcal{O}(n-1)$ is well defined. Then, there is a neighbourhood of 0 in $[0, l]$ on which we may write $A^{-1}(t)A'(t) = (\ln A)'(t)$, and this allows us to state:

THEOREM 4.4: *Given a motion $\Phi = \{\phi_t\}_{t\in I}$ along a curve $c(t)$ in M_λ^n, let $A(t)$ be the associated curve in $\mathrm{SO}(n-1)$. Let $\alpha(t) = \ln A(t)$, which is well defined on a neighbourhood of 0. Let us suppose that the curve $\alpha(t)$ is not contained in any hyperplane of $\mathcal{O}(n-1)$. If* volume($\mathcal{C}$) = volume($\mathcal{C}^P$), *then C_0 is contained in a geodesic sphere of P_0.*

If $\mathcal{C}_0$ is compact without boundary, then, for any dimension $n \geq 3$, $\mathcal{C}_0$ is a geodesic sphere of P_0 with centre at $c(0)$.

Proof: From Lemma 4.3 and the above remark, it is enough to show that the assumptions of the theorem imply the existence of points $t_2, \ldots, t_{n-1}$ satisfying the condition (4.5) for every $N(0)$. But, if there are no such points, there is a $N(0)$ such that the biggest integer k such that there are $\alpha'(t_2)N(0), \ldots, \alpha'(t_k)N(0)$ linearly independent is $\leq n-2$. Then, for every $t \in I$, $\alpha'(t)N(0)$ is a linear combination of $\alpha'(t_2)N(0), \ldots, \alpha'(t_k)N(0)$; so the curve $\alpha(t)N(0)$ is contained in an affine subspace of $T_{c(0)}P_0$ of dimension $k-1 \leq n-3$. Therefore, for every t, the vectors $\alpha'(t)N(0), \ldots, \alpha^{(k)}(t)N(0)$ are linearly dependent.

Let us write the matrices of $\mathcal{O}(n-1)$ using an orthonormal basis of $T_{c(0)}P_0$ of the form $\{N(0), e_2, \ldots, e_{n-1}\}$. Let us identify $\mathcal{O}(n-1)$ with $\mathbb{R}^{(n-1)(n-2)/2}$ by

$$\begin{pmatrix} 0 & -a_{23} & \ldots & -a_{2n} \\ a_{23} & 0 & \ldots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n} & a_{3n} & \ldots & 0 \end{pmatrix} \mapsto (a_{23}, \ldots, a_{2n}, a_{34}, \ldots, a_{n-1\ n});$$

then $\alpha(t)N(0) = (0, a_{23}(t), \ldots, a_{2n}(t))$ can be considered as the image $(a_{23}(t), \ldots, a_{2n}(t), 0, \ldots, 0)$ of $\alpha(t)$ by the natural projection π from $\mathbb{R}^{(n-1)(n-2)/2}$ onto the subspace $\mathbb{R}^{n-2} \times \{0\}$. Then, the condition written in the above paragraph says that the projection of the curve $\alpha(t)$ on the $(n-2)$-plane $\mathbb{R}^{n-2} \times \{0\}$ has its first $k \leq n-2$ derivatives linearly dependent; so this projection is a curve contained in a $(n-3)$-dimensional subspace Π, therefore $\alpha(t)$ is contained in the subspace $\pi^{-1}(\Pi) = \Pi \times \mathbb{R}^{(n-2)(n-3)/2}$ of dimension $\leq ((n-1)(n-2)/2) - 1$, contradicting the assumptions. ∎

COROLLARY 4.5: *If $n = 3$, and $\mathcal{C}$ is obtained by a motion Φ of $\mathcal{C}_0$ along a curve $c(t)$, with $c(0)$ the centre of mass of $\mathcal{C}_0$,* volume$(\mathcal{C})$ = volume$(\mathcal{C}^P)$ *implies that Φ is a parallel motion or that $\mathcal{C}_0$ is a circle, perhaps without one point.*

Proof: If $n = 3$, $\mathcal{O}(n-1) = \mathcal{O}(2)$ is isomorphic to $\mathbb{R}$. Then, the condition on α is just that it not be the constant map 0; but this means that $A(t)$ is not the constant map Id, that is, that $\varphi_t \neq \varphi_t^P$, i.e., that Φ is not a parallel motion, as claimed. ∎

Remark 4.6: The family of C^∞ curves $A(t)$ in SO$(n-1)$ satisfying the conditions of Theorem 4.4 is generic, that is, it contains an open and dense set in the family of C^∞ curves in SO$(n-1)$ with the Whitney's topology. It can be proved following standard arguments (cf. [NB, Th. 2.1], [Hi, pages 60 and 80] and [Wa, page 758]).

Remark 4.7: When $n = 3$, Theorem 4.4 says that parallel motion is unique giving the minimum of volume$(\mathcal{C})$ if $\mathcal{C}_0$ is not contained in a geodesic sphere. When $n \geq 4$, we lose uniqueness, and the best result that we may have is the genericity of the motions which do not give the minimum. In fact, the following is an easy example showing that the hypothesis of Theorem 4.4 on the motion is necessary.

In $\mathbb{R}^n$, let $\mathcal{C}_0$ be the cylinder $S^{p-2} \times J^{n-p}$ of $T_{c(0)}P_0$ with centre at $c(0)$, where $J =]-\epsilon, \epsilon[$ and S^{p-2} is a euclidean sphere of radius ϵ, with ϵ small enough in order that $\mathcal{C}_0$ be contained in the open set U_0 defined at the end of section 2.

Let $R(t)$ be a non-constant smooth curve in $\mathrm{SO}(p-1)$ such that $R(0) = \mathrm{Id}$. Let $E_2(t), \ldots, E_n(t)$ be a D-parallel frame along $c(t)$ such that $E_2(0), \ldots, E_p(0)$ generates the $(p-1)$-dimensional subspace of $T_{c(0)}P_0$ where S^{p-2} is contained. We define the motion Φ by

$$\varphi_t(E_i(0)) = \begin{cases} R(t)E_i(t) & \text{if } 2 \leq i \leq p, \\ E_i(t) & \text{if } p+1 \leq i \leq n. \end{cases}$$

The unit vector ξ_0 at $(u,a) \in S^{p-2} \times J^{n-p}$ is $\xi_0 = u/\epsilon$, and

$$\xi_t = \frac{1}{\epsilon} R(t) u.$$

The vector $N(0)$ corresponding to (u,a) is $N(0) = (u,a)/\sqrt{\epsilon^2 + |a|^2}$; then

$$\begin{aligned}\frac{DN(t)}{dt} = \frac{D\varphi_t N(0)}{dt} &= \frac{D}{dt}\Big(\sum_{i=2}^{p} N^i(0)R(t)E_i(t) + \sum_{i=p+1}^{n} N^i(0)E_i(t)\Big) \\ &= \frac{1}{\sqrt{\epsilon^2+|a|^2}} R'(t)u.\end{aligned}$$

So

$$\Big\langle \tau_t \frac{DN(t)}{dt}, \xi_t \Big\rangle = \frac{1}{\epsilon\sqrt{\epsilon^2+|a|^2}} \langle R'(t)u, R(t)u\rangle = 0.$$

Therefore, $\mathrm{volume}(\mathcal{C}) = \mathrm{volume}(\mathcal{C}^P)$ for this $\mathcal{C}_0$, which is not contained in a sphere of $T_{c(0)}P_0$.

§5. The role of curves

In this section we change the viewpoint. Instead, to consider a curve and to study the family of motions along it, we consider a motion well defined along a curve (a Frenet motion) and study this motion along a family of curves.

THEOREM 5.1: *Let $c(t)$ be a curve in M^n_λ such that*

(5.1) *the functions $k_2(t), \ldots, k_{n-1}(t)$ are linearly independent.*

Let $\mathcal{C}_0$ be a hypersurface of P_0 with centre of mass at $c(0)$. If $\mathrm{volume}(\mathcal{C}^F) = \mathrm{volume}(\mathcal{C}^P)$, *then $\mathcal{C}_0$ is contained in a geodesic sphere of P_0 with centre at $c(0)$.*

If $n = 3$, then $\mathcal{C}_0$ is a geodesic circle (perhaps without one point). If $\mathcal{C}_0$ is compact without boundary, then, for any dimension $n \geq 3$, $\mathcal{C}_0$ is a geodesic sphere of P_0.

Proof: It will be enough to see that, for a Frenet motion, if $\mathrm{volume}(\mathcal{C}^F) = \mathrm{volume}(\mathcal{C}^P)$, then the conditions (5.1) and (4.3) are equivalent.

Let us denote

$$v(t) := \sum_{i=2}^{n}(-N_{i+1}(0)k_i(t) + N_{i-1}(0)k_{i-1}(t))\overline{f}_i(0).$$

For a Frenet motion we have $N_i(t) = N_i(0)$ and $N_i'(t) = 0$. Then, it follows from (3.7) that condition (4.3) is equivalent to the existence of $n-2$ points $t_2, \dots, t_{n-1}$ satisfying

(5.2) the vectors $v_2 := v(t_2), \dots, v_{n-1} := v(t_{n-1})$ are linearly independent.

Then the theorem will be proved once we see that, under the condition volume$(\mathcal{C}^F) =$ volume$(\mathcal{C}^P)$, (5.1) is equivalent to (5.2).

Condition (5.2) is equivalent to the rank of the matrix

$$\begin{pmatrix} -N_3k_2(t_2) & \dots & -N_{i+1}k_i(t_2) + N_{i-1}k_{i-1}(t_2) & \dots & N_{n-1}k_{n-1}(t_2) \\ \vdots & & \vdots & & \vdots \\ -N_3k_2(t_{n-1}) & \dots & -N_{i+1}k_i(t_{n-1}) + N_{i-1}k_{i-1}(t_{n-1}) & \dots & N_{n-1}k_{n-1}(t_{n-1}) \end{pmatrix}$$

being $n-2$.

But if we compute the minors of this $(n-2) \times (n-1)$ matrix, we obtain that all of them are, up to the sign, of the form

$$N_{i_1} \dots N_{i_{n-2}} \begin{vmatrix} k_2(t_2) & \dots & k_{n-1}(t_2) \\ \vdots & & \vdots \\ k_2(t_{n-1}) & \dots & k_{n-1}(t_{n-1}) \end{vmatrix}.$$

Then, except for the points where $N_i = 0$ for some $i \in \{2, \dots, n\}$, the condition (5.2) is satisfied if and only if

$$\begin{vmatrix} k_2(t_2) & \dots & k_{n-1}(t_2) \\ \vdots & & \vdots \\ k_2(t_{n-1}) & \dots & k_{n-1}(t_{n-1}) \end{vmatrix} \neq 0. \tag{5.3}$$

Then we have proved the equivalence between (5.3) and (5.2) except for the points with $N_i = 0$ (let us recall that $N(0) = \gamma_{x_0}'(0)$, then $N(0)$ depends on x_0). We claim that in every neighbourhood of one of these points, there is a point with $N_i \neq 0$ for every $i \in \{2, \dots, n\}$. In fact, if $N_i = 0$ at x and there is an open neighbourhood U of x such that for every $y \in U$ there is some i such that $N_i = 0$ at y, that is, U is contained in a union of coordinate geodesic hyperplanes of P_0. Then there is a point $z \in U$ with a neighbourhood $U_z \subset U$ which is an open set in some geodesic hyperplane $N_i = 0$ (if not, U will be contained in the intersection of two or more hyperplanes, then it will not be an open set of a hypersurface of P_0). Then, we may take U_z connected and with no intersection with the geodesic hyperplanes $N_j = 0$, $2 \le j \ne i$. Since $U_z \subset \mathcal{C}_0$, ξ_0 is a unit vector normal to U_z, and, since U_z is contained in the geodesic hyperplane $N_i = 0$, $\overline{f}_i(0)$ is also orthogonal to U_z, in which case

$$\text{on } U_z, \quad \xi_o = \overline{f}_i(0). \tag{5.4}$$

Let us see that this is not compatible with the condition (5.1). In fact, if (5.4) holds, then, using again (3.7), $N_i(t) = N_i(0)$, and the condition (4.4) (equivalent to volume($\mathcal{C}^F$) = volume($\mathcal{C}^P$)), we get

$$0 = \left\langle \tau_r \frac{DN}{dt}, \xi_t \right\rangle = \left\langle \frac{DN}{dt}, f_i(t) \right\rangle = -N_{i+1}(0)k_i(t) + N_{i-1}(0)k_{i-1}(t). \tag{5.5}$$

But, since U_z is open in $N_i = 0$, there is a $y \in U_z$ satisfying $N_j(y) \ne 0$ for every $j \ne i$; so at this point, (5.5) contradicts the hypothesis (5.1). Hence our claim is proved.

The equivalence between (5.2) and (5.3) proves, according to the proof of Lemma 4.3, that

$$\gamma'_{x_0}(\text{dist}(x_0, c(0))) = \pm\xi_0 \tag{5.6}$$

holds except for the points with $N_i = 0$. Then, by continuity, the equality (5.6) holds everywhere, and we have that condition (5.3) implies that $\mathcal{C}_0$ is contained in a geodesic sphere of P_0 with centre at $c(0)$.

Now, we shall finish by showing that (5.3) is equivalent to (5.1). In fact, (5.3) is equivalent to the linear independence of the vectors $\overrightarrow{k_i} = (k_i(t_2), \ldots, k_i(t_{n-1}))$, $2 \le i \le n-1$. By the continuity of the functions $k_i(t)$, this is equivalent to condition (5.1). ∎

It follows from this theorem that, in case $n = 3$, volume($\mathcal{C}^F$) = volume($\mathcal{C}^P$), for a Frenet motion along a curve $c(t)$ not contained in a plane implies that $\mathcal{C}_0$ is a circle of P_0. For $n = 4$, the analogous statement occurs when the quotient k_3/k_2 is not constant.

Remark 5.2: Again, the family of curves satisfying (5.1) is generic.

Remark 5.3: Condition (5.1) of Theorem 5.1 is necessary. It is easy to find examples of $\mathcal{C}_0$ not contained in a geodesic sphere of P_0 and such that volume($\mathcal{C}^F$) = volume($\mathcal{C}^P$) when $c(t)$ does not satisfy (5.1). For instance, in $\mathbb{R}^4$, let $c(t)$ be any curve with $k_3/k_2 = k$ constant. Then, for any $N(t) = \varphi_t^F(N(0)) = \sum_{i=2}^4 N_i f_i(t)$,

$$\frac{DN}{dt}(t) = k_2\left\{-N_3 f_2(t) + (N_2 - kN_4) f_3(t) + kN_3 f_4(t)\right\}.$$

Let $\mathcal{C}_0$ be defined as the set of points in P_0 satisfying an equation of the form $g(x,y,z)=\epsilon$, where x,y,z are the coordinates of P_0 in the basis $f_2(0)$, $f_3(0)$, $f_4(0)$. We take ϵ small enough in order to have $\mathcal{C}_0$ intersected with a ball of P_0 with centre at $c(0)$ of adequate radius, not empty, and contained in the open set U_0 described at the end of section 2. We shall still denote by $\mathcal{C}_0$ this intersection. In this situation, condition (4.4) is equivalent to

$$-y\frac{\partial g}{\partial x}+(x-kz)\frac{\partial g}{\partial y}+ky\frac{\partial g}{\partial z}=0.$$

Among others, a solution of this equation is

$$g(x,y,z)=y^2+\frac{1}{1+k^2}(x-kz)^2,$$

which defines a cylinder $\mathcal{C}_0$ in P_0 satisfying volume$(\mathcal{C}^F)$ = volume$(\mathcal{C}^P)$ because (4.4) holds.

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